

Math 122 / Problem Set 2

Written problems due Monday, October 3

Monday, September 26

1. Let $\varphi : G \rightarrow G'$ be a group homomorphism, and let $x \in G$ be an element of order r . What can you say about the order of $\varphi(x)$?
2. Let $\varphi : G \rightarrow G'$ be a surjective homomorphism.
 - (a) Assume that G is abelian. Prove that G' is abelian.
 - (b) Assume that G is cyclic. Prove that G' is cyclic.
3. Prove that the set $\text{Aut } G$ of automorphisms of a group G forms a group, the law of composition being composition of functions.
4. Determine $\text{Aut } G$ where G is a cyclic group of order 10.

Reading: Artin §§2.5, 2.6

Wednesday, September 28

5. Let $x \sim y$ denote the relation “ x is conjugate to y .”
 - (a) Prove that \sim is an equivalence relation.
 - (b) Describe the elements a whose conjugacy class (= equivalence class) consists of the element a alone.
6. Let $\varphi : G \rightarrow G'$ be a homomorphism. Prove that the fibres of φ are the cosets of $N = \ker \varphi$.
7. Prove that every group whose order is a power of a prime p contains an element of order p .
8. Prove that every subgroup of index 2 is normal.

Reading: Artin §§2.7, 2.8

Friday, September 30

9. Let G and G' be finite groups whose orders have no common factor. Prove that the only homomorphism $\varphi : G \rightarrow G'$ is the trivial one $\varphi(x) = 1$ for all x .
10. Let G be a finite group whose order is a product of two integers $n = ab$. Let H, K be subgroups of G of orders a and b respectively. Assume that $H \cap K = \{1\}$. Prove that $HK = G$. Is G isomorphic to the product group $H \times K$?

Reading: Artin §2.9